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ABSTRACT

A compendium of thirty previously published open problems in computational geometry is presented.

The computational geometry community has made many advances in the relatively short (quarter-century) of the field's existence. Along the way researchers have engaged with a number of problems that have resisted solution. We gather here a list of open problems in computational geometry (and closely related disciplines) which together have occupied a sizable portion of the community's efforts over the last decade or more. We make no claim to comprehensiveness, only that were all these problems to be solved, the field would be greatly advanced. All the problems have appeared in earlier publications, but we believe all remain open as stated. We present them in condensed form, without always defining every technical term, but in each case providing at least one reference for further investigation. Our list consists of predominantly theoretical questions for which the problem can be succinctly stated and the measure of success is clear. We do not attempt here to list the wealth of important problems in applied and experimental computational geometry now being addressed by the community as it responds to the application-driven need for practical geometric algorithms; we hope that an ongoing project to compile a more comprehensive list will address this omission. We encourage correspondence to correct, extend, and update a Web version of this list.[a]

1. Can a *minimum weight triangulation* of a planar point set — one minimizing

[a]http://cs.smith.edu/ orourke/TOPP/

the total edge length—be found in polynomial time? This problem is one of the few from Garey and Johnson[39] whose complexity status remains unknown. The best approximation algorithms achieve a (large) constant times the optimal length[49]; good heuristics are known.[32] If Steiner points are allowed, again constant-factor approximations are known,[35,29] but it is open to decide even if a minimum-weight Steiner triangulation exists (the minimum might be approached only in the limit).

2. What is the maximum number of combinatorial changes possible in a Euclidean Voronoi diagram of a set of n points each moving along a line at unit speed in two dimensions? The best lower bound known is quadratic, and the best upper bound is cubic.[62] If the speeds are allowed to differ, the upper bound remains essentially cubic.[4] The general belief is that the complexity should be close to quadratic; Chew[27] showed this to be the case if the underlying metric is L_1 (or L_∞).

3. What is the combinatorial complexity of the Voronoi diagram of a set of lines (or line segments) in three dimensions? This problem is closely related to the previous problem, because points moving in the plane with constant velocity yield straight-line trajectories in space-time. Again, there is a gap between a lower bound of $\Omega(n^2)$ and an upper bound that is essentially cubic[64] for the Euclidean case (and yet is quadratic for polyhedral metrics[19]). A recent advance shows that the "level sets" of the Voronoi diagram of lines, given by the union of a set of cylinders, indeed has near-quadratic complexity.[12]

4. What is the complexity of the union of "fat" objects in $\mathbb{R}^3$? The Minkowski sum of polyhedra of n vertices has complexity $O(n^{2+\epsilon})$,[10] as does the union of n congruent cubes.[60] It is widely believed the same should hold true for *fat* objects, those with a bounded ratio of circumradius to inradius, as in does in $\mathbb{R}^2$.[38]

5. Can the Euclidean *minimum spanning tree* (MST) of n points in $\mathbb{R}^d$ be computed in time close to the lower bound of $\Omega(n \log n)$?[40] An MST of a connected graph can be computed in time nearly linear in the number of edges,[25] but this is quadratic in the number of vertices n for geometric graphs. And indeed the best upper bounds for the Euclidean MST approach quadratic for large d, e.g.,Ref. [28]. This problem is intimately related to the bichromatic closest pair problem.[2]

6. What is the complexity of computing a minimum-cost Euclidean matching for $2n$ points in the plane? An algorithm that achieves the minimum and runs in nearly $O(n^{2.5})$ time has long been available.[70] Recently Arora showed how to achieve a $(1+\epsilon)$-approximation in $n(\log n)^{O(1/\epsilon)}$ time.[8]

7. What is the maximum number of k-sets? (Equivalently, what is the maximum complexity of a k-level in an arrangement of hyperplanes?) For a given set P of n points, $S \subset P$ is a *k-set* if $|S| = k$ and $S = P \cap H$ for some open

halfspace H. Even for points in two dimensions the problem remains open: The maximum number of k-sets as a function of n and k is known to be $O(nk^{1/3})$ by a recent advance of Dey,[30] while the best lower bound is only slightly superlinear.[67]

8. Is linear programming strongly polynomial? It is known to be weakly polynomial, exponential in the bit complexity of the input data.[46,45] Subexponential time is achievable via a randomized algorithm.[56] In any fixed dimension, linear programming can be solved in strongly polynomial linear time (linear in the input size).[33,51]

9. Can every convex polyhedron be cut along its edges and unfolded flat to a single, nonoverlapping, simple polygon? The answer is known to be NO for nonconvex polyhedra,[14] but has been long open for convex polyhedra.[65,58]

10. Is there a deterministic, linear-time polygon triangulation algorithm significantly simpler than that of Chazelle?[23] Simple randomized algorithms that are close to linear-time are known,[63] and a recent randomized linear-time algorithm[5] avoids much of the intricacies of Chazelle's algorithm. Relatedly, is there a simple linear-time algorithm for computing a shortest path in a simple polygon, without first applying a more complicated triangulation algorithm?

11. Can the class of *3-SUM hard* problems[41] be solved in subquadratic time? These problems can be reduced from the problem of determining whether, given three sets of integers, A, B, and C with total size n, there are elements $a \in A$, $b \in B$, and $c \in C$ such that $a + b = c$. Many fundamental geometric problems fall in this class; e.g., computing the area of the union of n triangles.

12. Can a planar convex hull be maintained to support both dynamic updates and queries in logarithmic time? Recently the $O(\log^2 n)$ barrier was broken with a $O(\log^{1+\epsilon} n)$ update- and $O(\log n)$ query-time structure,[26] and the update time further improved to $O(\log n \log\log n)$ in Ref. [17].

13. Is there an $O(n)$-space data structure that supports $O(\log n)$-time point-location queries in a three-dimensional subdivision of n faces? Currently $O(n \log n)$ space and $O(\log^2 n)$ queries are achievable.[66]

14. Is it possible to construct a ***binary space partition*** (BSP) for n disjoint line segments in the plane of size less than $\Theta(n \log n)$? The upper bound of $O(n \log n)$ was established by Paterson and Yao.[61] Recently Tóth[68] improved the trivial $\Omega(n)$ lower bound to $\Omega(n \log n / \log\log n)$. Can the remaining gap be closed?

15. What is the best output-sensitive convex hull algorithm for n points in $\mathbb{R}^d$? The lower bound is $\Omega(n \log f + f)$ for f facets (the output size). The best upper bound to date is $O(n \log f + (nf)^{1-\delta} \log^{O(1)} n)$ with $\delta = 1/(\lfloor d/2 \rfloor + 1)$,[24] which is optimal for sufficiently small f.

16. Can the number of simple polygonalizations of a set of n points in the plane be computed in polynomial time? Certain special cases are known (e.g., monotone simple polygonalizations[72]), but the general problem remains open. The problem is closely related to that of generating a "random" instance of a simple polygon on a given set of vertices. Heuristic methods are known and implemented.[6]

17. Given a visibility graph G and a Hamiltonian circuit C, determine in polynomial time whether there is a simple polygon whose vertex visibility graph is G, and whose boundary corresponds to C. The problem is not even known to be in NP,[57] although it is for "pseudo-polygon" visibility graphs.[59]

18. When a collection of disks are pushed closer together, so that no distance between two center points increases, can the area of their union increase? It seems the answer is NO, but this has only been settled in the continuous-motion case.[18] The corresponding question for intersection area decrease is similarly unresolved.[20]

19. What is the complexity of the ***vertical decomposition*** of n surfaces in $\mathbb{R}^d$, $d \geq 5$? The lower bound of $\Omega(n^d)$ was nearly achieved up to $d = 3$,[11] but a gap remained for $d \geq 4$. A recent result[48] covers $d = 4$ and achieves $O(n^{2d-4+\epsilon})$ for general d, leaving a gap for $d \geq 5$.

20. What is the complexity of computing a spanning tree of a planar point set having minimum stabbing number? The ***stabbing number*** of a tree T is the maximum number of edges of T intersected by a line. Any set of n points in the plane has a spanning tree of stabbing number $O(\sqrt{n})$,[3,22,71] and this bound is tight in the worst case. However, nothing is known about the complexity of computing a spanning tree (or triangulation) of minimum stabbing number, exactly or approximately.

21. Can shortest paths among n obstacles in the plane, with a total of n vertices, be found in optimal $O(n + h \log h)$ time using $O(n)$ space? The only algorithm that is linear in n in time and space is quadratic in h[47]; $O(n \log n)$ time, using $O(n \log n)$ space, is known.[43] In three dimensions, the Euclidean shortest path problem among general obstacles is NP-hard, but its complexity remains open for some special cases, such as when the obstacles are disjoint unit spheres or axis-aligned boxes; see Ref. [53].

22. Can a minimum-link path among polygonal obstacles be found in subquadratic time? The best algorithm known requires essentially quadratic time in the worst case.[54] What is the complexity of computing minimum-link paths in three dimensions?

23. How many π-floodlights are always sufficient to illuminate any polygon of n vertices, with at most one floodlight placed at each vertex? An α-***floodlight*** is a light of aperture α. It was established in Ref. [34] that for any $\alpha < \pi$, there

is a polygon that cannot be illuminated with an α-floodlight at each vertex. When $\alpha = \pi$, $n-2$ lights (trivially) suffice. So it is of interest (as noted in Ref. [69)] to learn whether a fraction of n lights suffice for π-floodlights.

24. Can an n-vertex polygonal curve be simplified in time nearly linear in n? The goal is to compute a simplification that uses the fewest vertices of the original curve while approximating it according to some prescribed error criterion. Quadratic-time algorithms have been known for some time[21] and a recent algorithm achieves time $O(n^{4/3+\epsilon})$ for a certain error criterion.[13] In practice, the Douglas-Peucker algorithm is often used as a heuristic; it can be implemented to run in time $O(n \log n)$.[42]

25. How efficiently can one compute a polyhedral surface that is an ϵ-approximation of a given triangulated surface in $\mathbb{R}^3$? It is NP-hard to obtain the minimum-facet surface separating two nested convex polytopes,[31] but polynomial-time approximation algorithms are known[16,55,9] for this case, and for separating two terrain surfaces, achieving factors within $O(1)$ or $O(\log n)$ of optimal. However, no polynomial-time approximation results are known for general surfaces.

26. Given a sufficiently dense sample of points on a surface (technically, an ϵ-sample), reconstruct a surface homeomorphic to the original. This has recently been accomplished for smooth surfaces,[1] but remains open for surfaces with sharp edges and corners.

27. Can the interior of every simply connected polyhedron whose surface is meshed by an even number of quadrilaterals be partitioned into a hexahedral mesh compatible with the surface meshing?[15] It is known that a topological hexahedral mesh exists,[52,36] but despite the availability of software that extends quadrilateral surface meshes to hexahedral volume meshes, it is not known if all hexahedral cells have planar faces.

28. Is the ***flip graph*** connected for general-position points in $\mathbb{R}^3$? Given a set of n points in $\mathbb{R}^3$, the flip graph has a node for each tetrahedralization of the set. Two nodes are connected by an arc if there is a 2-to-3 or 3-to-2 "bistellar flip" of tetrahedra between the two simplicial complexes. In the plane, the flips correspond to convex quadrilateral diagonal switches; in $\mathbb{R}^3$, a 5-vertex convex polyhedron is "flipped" between two of its tetrahedralizations. In $\mathbb{R}^2$ the flip graph is connected, providing a basis for algorithms to iterate toward the Delaunay triangulation. A decade ago, several[37,44] asked whether the same holds in $\mathbb{R}^3$ (when no four points are coplanar), but the question remains unresolved.

29. Can every convex polytope in $\mathbb{R}^3$ be partitioned into tetrahedra such that the dual graph has a Hamiltonian path? Every convex polygon has such a ***Hamiltonian triangulation***, but not every nonconvex polygon does.[7] The

existence of a Hamiltonian path permits faster rendering on a graphics screen via pipelining.

30. We close with Conway's venerable thrackle conjecture, which remains open after more than thirty years. A *thrackle* is a planar drawing of a graph of n vertices by edges which are smooth closed curves between vertices, with the condition that any two edges intersect at exactly one point, and have distinct tangents there. Conway's conjecture is that the number edges cannot exceed n. Recently the upper bound was lowered from $O(n^{3/2})$ to $2n - 3$.[50] Conway offers a reward of $1,000 for a resolution.

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